Integral Global Optimization Method for Solution of Nonlinear Complementarity Problems

M. M. KOSTREVA and QUAN ZHENG
Department of Mathematical Sciences, Clemson University, Clemson, SC 29634-1907, U.S.A.

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Abstract. The mapping in a nonlinear complementarity problem may be discontinuous. The integral global optimization algorithm is proposed to solve a nonlinear complementarity problem with a robust piecewise continuous mapping. Numerical examples are given to illustrate the effectiveness of the algorithm.

Key words. Nonlinear programming, nonlinear complementarity problem, integral global optimization.

1. Introduction

Historically, the use of optimization methods to solve nonlinear complementarity problems has been obstructed by the fact that the solution of global optimization problems was required. In general, these global optimization problems involved constraint sets which were not convex, and did not always satisfy constraint qualifications. Sometimes the defining functions were not differentiable. The objective functions for such optimization approaches to complementarity were also difficult to handle and were neither concave nor convex. The depth of the technical difficulties resulting from all these factors has discouraged the research community from this line of thinking. However, recent progress in global optimization, now causes a re-examination of the problem. A new method of global optimization which is based on integration of functions has been developed [6–12]. From this fresh point of view, it is possible to handle the technical difficulties mentioned above and to resolve them in a systematic way. In this research we will investigate the solution of nonlinear complementarity problems via integral global minimization methods.

Some related work has been recently completed by Mangasarian and Solodov [4]. In their paper, the nonlinear complementarity problem is reformulated as an unconstrained minimization problem and then solved by local methods. Applying these methods from many starting points, they are often able to solve the nonlinear complementarity problem. However, with their approach it is quite possible that a suitable starting point will not be chosen and hence they will miss the solution to the nonlinear complementarity problem. They also assume the functions are differentiable in order to apply existing local methods of optimization. In the approach followed here, such assumptions are not necessary.

Let $f: R^n \to R^n$ be a given mapping, O an orthant in R^n. The complementarity problem associated with f is:

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$$\text{Find } x \in R^n \text{ such that } x \in O\,, \quad f(x) \in O^* = O\,, \quad \langle x, f(x)\rangle = 0\,, \tag{1.1}$$

where

$$\langle x, f(x)\rangle = x_1 f_1(x) + \cdots + x_n f_n(x)\,.$$

The mapping f is not necessarily assumed to be continuous. For instance, Habetler and Kostreva [2] consider problem (1.1) when f is a P-mapping. Recall that in [5] a mapping $f: R^n \to R^n$ is said to be a P-mapping on a set S if for all $x, y \in S$ with $x \neq y$, there exists an index $i = i(x, y)$ such that $(x_i - y_i)(f_i(x) - f_i(y)) > 0$. A P-mapping must be one-to-one, but need not be continuous.

Let $N = \{1, 2, \ldots, n\}$ and I^k, $k = 1, 2, \ldots, 2^n$ be subsets of N. Let $f: R^n \to R^n$ be a P-mapping on R^n. If for each $k = 1, \ldots, 2^n$ the mapping

$$f_i^k(x) = \begin{cases} f_i(x)\,, & i \in I^{(k)}\,, \\ x_i\,, & i \in N \backslash I^{(k)} \end{cases}$$

is a mapping from R^n *onto* R^n, then f is called a *nondegenerate P-mapping*.

The following theorem represents a quite general result for nonlinear complementarity problems, since the functions are not required to be differentiable or even continuous and the orthant of definition is left general. However, this level of generality is nevertheless compatible with an approach through the integral global optimization.

THEOREM 1.1. [2]. *Let*: $R^n \to R^n$ *be a nondegenerate P-mapping. Then for each* $O \subset R^n$, (1.1) *has a unique solution.*

The complementarity problem (1.1) can be formulated as the following minimization problem:

$$\min_{x \in S} g(x)\,, \tag{1.2}$$

where

$$g(x) = \langle x, f(x)\rangle \quad \text{and} \quad S = \{x \in R^n : x \in O, f \in O\}\,. \tag{1.3}$$

The problem (1.1) has solutions if and only if the global minimum value of (1.2) is equal to 0 and the set of global minimizers is the solution set of (1.1).

To solve (1.2), a nonsequential unconstrained minimization algorithm for finding the set of global minimizers of a constrained problem is proposed as follows:

ALGORITHM

Step 1: Take $c_0 > \min_{x \in S} g(x)$ and $\epsilon > 0$; take $\alpha_0 > 0$ sufficiently large and $\beta > 1.0$;

$$H_0 := \{x : g(x) + \alpha_0 p_S(x, \delta) \leq c_0\} \ ; \quad k := 0 \ ;$$

Step 2: Calculate the penalized mean value

$$c_{k+1} := \frac{1}{\mu(H_k)} \int_{H_k} [g(x) + \alpha_k p_S(x, \delta)] \, d\mu \ ;$$

with

$$H_k = \{x : g(x) + \alpha_k p_S(x, \delta) \leq c_k\} \ ;$$

Step 3: Calculate the penalized variance

$$v := \frac{1}{\mu(H_k)} \int_{H_k} [g(x) + \alpha_k p_S(x, \delta) - c_k]^2 \, d\mu \ ;$$

if $v > \epsilon$ then $\alpha_{k+1} = \alpha_k \cdot \beta$; $k := k + 1$; go to Step 2; otherwise, go to Step 4;
Step 4: $c^* \Leftarrow c_{k+1}$; $H^* \Leftarrow H_{k+1}$; Stop.

Here $\epsilon > 0$ is the accuracy requirement given in advance and $p_S(x, \delta)$ is a penalty function defined by (3.4) and (3.5).

As was discussed in [6, 7], a problem formulated with a nonrobust mapping may be numerically unapproximatable and unstable. Thus, we restrict ourselves to study the problem of a *robust piecewise continuous mapping f*. In the next section, we will review a few basic concepts of robust sets, mappings and the integral approach of minimization which we will use for further consideration. We will examine robust piecewise continuous mappings in Section 3. In Section 4, we will give numerical examples to illustrate the effectiveness of the algorithm.

2. Integral Global Minimization

In this section we will summarize several concepts and properties of the integral global minimization of robust discontinuous functions, which will be utilized in the following sections. For more details, see [8, 9, 12].

Let X be a topological space, a set D in X is said to be *robust* if

$$\text{cl}\, D = \text{cl}\,\text{int}\, D \ , \qquad (2.1)$$

where $\text{cl}\, D$ denotes the closure of D and $\text{int}\, D$ the interior of D.

A robust set consists of *robust points* of the set. A point $x \in D$ is said to be a robust point of D, if for each neighbourhood $N(x)$ of x, $N(x) \cap \text{int}\, D \neq \emptyset$. A set D is robust if and only if each point of D is a robust point of D. A point $x \in D$ is a robust point of D if and only if there exists a net $\{x_\lambda\} \subset \text{int}\, D$ such that $x_\lambda \to x$.

The interior of a nonempty robust set is nonempty. A union of robust sets is robust. An intersection of two robust sets may be nonrobust; but the intersection of an open set and a robust set is robust. A set D is robust if and only if $\partial D = \partial \operatorname{int} D$, where $\partial D = \operatorname{cl} D \backslash \operatorname{int} D$ denotes the boundary of D. A robust set can be represented as a union of an open set and a *nowhere dense* set.

A function $f: X \to R^n$ is said to be *upper robust* if the set

$$F_c = \{x: f(x) < c\} \tag{2.2}$$

is robust for each real number c.

An upper semicontinuous function is upper robust since (2.2) is open for each c. If X is a complete metric space, then the set of points of discontinuity (continuity) of an upper robust function is of first (second) category.

A function f is upper robust if and only if it is upper robust at each point; f is upper robust at a point x if $x \in F_c$ implies x is robust to F_c.

EXAMPLE 2.1. An example of a non upper robust function on R^1 is

$$f(x) = \begin{cases} 0, & x = 0, \\ 1, & x \neq 0. \end{cases}$$

f is nonrobust at $x = 0$.

In [6], robust and approximatable mappings are studied. Let X and Y be topological spaces. A mapping $f: X \to Y$ is said to be *robust* if for each open set $G \subset Y$, $f^{-1}(G)$ is a robust set in X.

The following example shows that a P-mapping may be nonrobust.

EXAMPLE 2.2. Let $f = (f_1, f_2): R^2 \to R^2$ be defined as follows:

$$f_1(x_1, x_2) = \begin{cases} x_1 + 1, & x_1 > 0 \text{ and } \forall x_2, \\ 0.1, & x_1 = 0 \text{ and } \forall x_2, \\ x_1 - 1, & x_1 < 0 \text{ and } \forall x_2 \end{cases}$$

and

$$f_2(x_1, x_2) = x_2 + 0.5, \quad \forall x_1 \text{ and } x_2.$$

It is easy to verify that the mapping f is a P-mapping. For this mapping the complementarity problem (1.1) has a solution $x = (0, 0)^T$ and $y = (0.1, 0.5)^T$. However, f is nonrobust. Take $G = (-0.5, 0.5) \times (0, 1)$, then $f^{-1}(G) = \{0.1\} \times (-0.5, 0.5)$. $f^{-1}(G)$ is a nonrobust set in R^2.

Suppose C is the set of points of continuity of f. f is said to be *approximatable* iff C is dense in X and for each $\bar{x} \in X$, there exists a net $\{x_\alpha\} \subset C$ such that

$$\lim_\alpha x_\alpha = \bar{x} \quad \text{and} \quad \lim_\alpha f(x_\alpha) = f(\bar{x}).$$

An approximatable mapping is robust. If X is a Baire space and Y satisfies the second axiom of countability, then a mapping is robust if and only if it is approximatable.

In order to investigate a minimization problem with an integral approach, a special class of measure spaces, which are called Q-measure spaces, should be examined.

Let X be a topological space, Ω a σ-field of subsets of X and μ a measure on Ω. A triple (X, Ω, μ) is called a Q-measure space iff

(i) Each open set in X is measurable;
(ii) The measure $\mu(G)$ of each nonempty open set G in X is positive: $\mu(G) > 0$;
(iii) The measure $\mu(K)$ of a compact set K in X is finite.

The n-dimensional Lebesgue measure space (R^n, Ω, μ) is a Q-measure space; a nondegenerate Gaussian measure μ on a separable Hilbert space H with Borel sets as measurable sets constitutes an infinite dimensional Q-measure space. A specific optimization problem is related to a specific Q-measure space which is suitable for consideration in this approach.

Once a measure space is given we can define integration in a conventional way.

Since the interior of a nonempty open set is nonempty, *the Q-measure of a measurable set containing a nonempty robust set is always positive*. This is an essential property we need in the integral approach of minimization. Hence, the following assumptions are usually required:

ASSUMPTION (A′). *f is Q-measurable.*

ASSUMPTION (R). *f is upper robust and bounded below on S.*

ASSUMPTION (M). *(X, Ω, μ) is a Q-measure space.*

In the following application, we need a lemma.

LEMMA 2.1. *Suppose that the conditions (A'), (M) and (R) hold. If $c > c^* = \min_{x\in S} f(x)$, then*

$$\mu(H_c \cap S) > 0 .$$

Suppose that the assumptions (A′), (M) and (R) hold, and $c > c^* = \inf_{x\in S} f(x)$. We define the mean value, variance, modified variance and m-th moment (centered at a), respectively, as follows:

$$M(f, c; S) = \frac{1}{\mu(H_c \cap S)} \int_{H_c \cap S} f(x)\, d\mu ,$$

$$V(f, c; S) = \frac{1}{\mu(H_c \cap S)} \int_{H_c \cap S} (f(x) - M(f, c; S))^2\, d\mu ,$$

$$V_1(f, c; S) = \frac{1}{\mu(H_c \cap S)} \int_{H_c \cap S} (f(x) - c)^2 \, d\mu ,$$

$$M_m(f, c; a; S) = \frac{1}{\mu(H_c \cap S)} \int_{H_c \cap S} (f(x) - a)^m \, d\mu , \quad m = 1, 2, \ldots$$

By Lemma 2.1, they are well defined. These definitions can be extended to the case $c \geqslant c^*$ by a limit process. For instance,

$$M_m(f, c; a; S) = \lim_{c_k \downarrow c} \frac{1}{\mu(H_{c_k} \cap S)} \int_{H_{c_k} \cap S} (f(x) - a)^m \, d\mu , \quad m = 1, 2, \ldots$$

The limits exist and are independent of the choice of $\{c_k\}$. The extended concepts are well defined and consistent with the above definitions.

With these concepts we characterize the global optimality as follows:

THEOREM 2.1. *Under the assumptions* (A'), (M) *and* (R), *the following statements are equivalent:*

(i) $x^* \in S$ *is a global minimizer of* f *over* S *and* $c^* = f(x^*)$ *is the global minimum value*;
(ii) $M(f, c^*; S) = c^*$ (*the mean value condition*);
(iii) $V(f, c^*; S) = 0$ (*the variance condition*);
(iv) $V_1(f, c^*; S) = 0$ (*the modified variance condition*);
(v) $M_m(f, c^*; c^*; S) = 0$, *for one of positive integers* $m = 1, 2, \ldots$ (*the higher moment conditions*).

3. Robust Piecewise Continuous Mappings

In this section we will examine basic properties of robust piecewise continuous mapping and formulate a nonlinear complementarity problem as an unconstrained minimization by using a discontinuous penalty function.

DEFINITION 3.1. Suppose S is a robust set of a topological space X. If there is a family of robust sets $\{V_\lambda\}$, $\lambda \in \Lambda$ such that

$$S = \bigcup_{\lambda \in \Lambda} V_\lambda \quad \text{and} \quad \forall \alpha \neq \lambda , \quad V_\alpha \cap V_\lambda = \emptyset , \tag{3.1}$$

then $\{V_\lambda\}$ is called a robust partition of S. Suppose $\{U_\alpha\}$, $\alpha \in A$ is another robust partition of S. If for each V_λ there is U_α such that $U_\alpha \subset V_\lambda$, then $\{U_\alpha\}$, $\alpha \in A$ is called a robust subpartition of $\{V_\lambda\}$.

DEFINITION 3.2. Let X and Y be two topological spaces, S a robust set in X. A mapping $f: S \subset X \to Y$ is said to be robust piecewise continuous iff there exists a robust partition $\{V_\lambda\}$ of S, such that for any $\lambda \in \Lambda$, the restriction of f to V_λ is continuous.

PROPOSITION 3.1. *Let X and Y be topological spaces, and $f: X \to Y$ a mapping. If f is robust piecewise continuous with a robust partition $\{V_\lambda\}$ of a robust set S, then it is robust.*

Proof. Suppose $G \subset Y$ is an open set, we will prove that $f^{-1}(G) \cap S$ is a robust set. Indeed,

$$f^{-1}(G) \cap S = f^{-1}(G) \cap \bigcup_\lambda V_\lambda = \bigcup_\lambda (f^{-1}(G) \cap V_\lambda).$$

The intersection of the open set $f^{-1}(G)$ and the robust set V_λ is robust, and the union of robust sets is robust. □

REMARK 3.1. Note that if in the above definition the partition of S is not required to be *robust*, a piecewise continuous mapping may be non robust.

The class of robust piecewise continuous mappings with the *same* robust partition has some desirable properties.

PROPOSITION 3.2. *Let X be a topological space, Y a linear topological space, and $f, g: X \to Y$ mappings. If f and g are robust piecewise mappings with the same robust partition, then for real numbers α and β, $\alpha \cdot f + \beta \cdot g$ is also a robust piecewise continuous mapping.*

Proof. Suppose f and g are robust piecewise continuous with a robust partition $\{V_\lambda\}$. For each give robust set V_λ in the partition, f and g are continuous on it; so is the function $\alpha \cdot f + \beta \cdot g$. Hence, $\alpha \cdot f + \beta \cdot g$ is robust piecewise continuous with the partition $\{V_\lambda\}$.

The following two propositions can be proved similarly.

PROPOSITION 3.3. *Let X be a topological space and $f, g: X \to R^1$ functions. If f and g are robust piecewise continuous with the same robust partition, then $f \cdot g$, $f/g (g \neq 0)$, $\max(f, g)$, $\min(f, g)$ and $|f|$ are also robust piecewise continuous.*

PROPOSITION 3.4. *Let X be a topological space, then $f = (f_1, \ldots, f_n)^T: X \to R^n$ is a robust piecewise continuous mapping if and only if each of the component functions f_i, $i = 1, \ldots, n$ is robust piecewise continuous with the same robust partition.*

For the complementarity problem (1.1), the feasible set is

$$S = \{x \in R^n : x \in O, f \in O\}. \tag{3.2}$$

We assume S and S^c are robust and $X = \{S, S^c\}$ has a robust subpartition, and assume that f is robust piecewise continuous with respect to this robust subpartition.

We can use a discontinuous penalty function to formulate the constrained minimization problem (1.2) as an unconstrained one, where we assume that $X = R^n$ and $O = \{x = (x_1, \ldots, x_n)^T : x_i \geqslant 0, i = 1, \ldots, n\}$:

$$\min[\langle x, f(x)\rangle + \alpha p_S(x, \delta)], \tag{3.3}$$

where $p_S(x, \delta)$ is defined as follows:

$$p_S(x, \delta) = \begin{cases} 0, & x \in S, \\ \delta + d(x), & x \in S^c, \end{cases} \tag{3.4}$$

where $\delta > 0$ is given and

$$d(x) = \sum_{i=1}^{n} [|\min(x_i, 0)| + |\min(f_i(x), 0)|]. \tag{3.5}$$

Note that in the above definition we relax the requirement of continuity from the traditional definition [1, 3] as we wish to utilize discontinuous penalty functions.

DEFINITION 3.3. A function $p(x)$ on X is a penalty function for a constraint set S if

(i) $p(x) = 0$ if $x \in S$;
(ii) $\inf_{x \notin S_\beta} p(x) > 0$, where $S_\beta = \{u : \|u - v\| \leqslant \beta, \forall v \in S\}$ and $\beta > 0$.

REMARK 3.2. It is expected that the penalty will be increasing when the distance of a point X to the constraint set S is getting larger. We replace the traditional property

$$p(x) > 0, \quad \text{if} \quad u \notin S$$

by (ii).

DEFINITION 3.4. A penalty function p for the constraint set S is exact for a minimization problem

$$\min_{x \in S} g(x) \tag{3.6}$$

if there is a real number $\alpha_0 > 0$ such that for each $\alpha \geqslant \alpha_0$ we have

$$\min_{x \in X} \{g(x) + \alpha p(x)\} = \min_{x \in S} g(x) = c^* \tag{3.7}$$

and

$$\{x: g(x) + \alpha p(x) = c^*\} = \{x \in S: g(x) = c^*\} = H^* . \tag{3.8}$$

PROPOSITION 3.5. *Suppose $X = R^n$ and f is robust piecewise continuous with a robust subpartition of $\{S, S^c\}$, then for each $\alpha > 0$ and $\delta > 0$ the penalized function*

$$\langle x, f(x)\rangle + \alpha p_S(x, \delta) \tag{3.9}$$

is a piecewise robust continuous function.

Proof. Suppose $\{V_\lambda\}$ is the robust partition with which f is robust piecewise continuous. Then the component functions f_i, $i = 1, \ldots, n$ are robust piecewise with it. Thus,

$$\langle x, f(x)\rangle = x_1 f_1(x) + \cdots + x_n f_n(x)$$

is robust piecewise continuous. (The functions $|\min(x_i, 0)|$ and $|\min(f_i, 0)|$, $i = 1, \ldots, n$ are robust piecewise continuous, so is the function $\delta + d(x)$.) Since, $\{V_\lambda\}$ is a subpartition of $\{S, S^c\}$, thus, the penalty function $p_S(x, \delta)$ is robust piecewise continuous with $\{V_\lambda\}$. As the sum of $\langle x, f(x)\rangle$ and $\alpha p_S(x, \delta)$, the penalized function (3.9) is robust piecewise with $\{V_\lambda\}$. □

When we use the integral approach to deal with minimization problems, a Q-measure space is used. Then we require that each set V_λ in the robust partition is measurable in the given Q-measure space. A robust partition $\{V_\lambda\}$ is called a *measurable robust partition* if each set in the partition is measurable. If $\{V_\lambda\}$ is a measurable robust partition, then a robust piecewise continuous function with this partition is measurable.

Observe that the conditions (A′), (M) and (R) hold for problem (3.3). The penalty function (3.4) with (3.5) is exact [see, 10, 11]. We can use integral minimization algorithms to solve the unconstrained problem (3.3).

Return to the *algorithm* in Section 1. Let $\epsilon = 0$ in the algorithm. It may stop in a finite number of steps or we obtain a decreasing sequence

$$c_0 > c_1 > \cdots > c_k > c_{k+1} > \cdots \geqslant c^* \tag{3.10}$$

and a monotone sequence of sets

$$H_0 \supset H_1 \supset \cdots \supset H_k \supset H_{k+1} \supset \cdots . \tag{3.11}$$

The limits of these sequences exist. Let

$$c^* = \lim_{n \to \infty} c_k \tag{3.12}$$

and

$$H^* = \lim_{k \to \infty} H_k = \bigcap_{k=1}^{\infty} H_k . \tag{3.13}$$

The following theorems can be proved by applying Theorem 2.1 (see [12], Theorem 5.3.3).

THEOREM 3.1. *Under the assumptions* (A'), (M) *and* (R), *the limit* c^* *of* (3.12) *is the global minimum value and the limit* H^* *of* (3.13) *is the set of global minimizers of* g *over* S.

COROLLARY 3.1. *Under the assumptions of Theorem* 3.1, *if* f *is a nondegenerate P-mapping, then complementarity problem* (1.1) *is solvable by the integral optimization method.*

Note that the errors at each iteration in the algorithm are not accumulated. Suppose we calculate $c_1 = M(g, c_0; S)$ with an error Δ_1 and obtain $d_1 = c_1 + \Delta_1$; then calculate $c_2' = M(f, d_1; S)$ with an error Δ_2, and obtain $d_2 = c_2' + \Delta_2$, and so on. In general, we have

$$c_k' = M(g, d_{k-1}; S) \quad \text{and} \quad \Delta_k = d_k - c_k' , \quad k = 1, 2, \ldots \tag{3.14}$$

and obtain a decreasing sequence $\{d_k\}$. Let

$$d = \lim_{k\to\infty} d_k . \tag{3.15}$$

THEOREM 3.2. *Under the assumptions of Theorem* 3.1, d *is the global minimum value of* g *over* S *if and only if*

$$\lim_{k\to\infty} \Delta_k = 0 . \tag{3.16}$$

The algorithm has been implemented by a properly designed Monte-Carlo method. At each iteration we need to find: (1) A level set; (2) a mean value and (3) a modified variance (multi-dimensional integrations). Monte-Carlo technique can handle higher dimensional integration with lower accuracy:

$$\delta \approx \frac{C_p}{\sqrt{N}} \sigma ,$$

where N is the number of sample points and σ^2 is the variance. $\delta \to 0$ as $\sigma \to 0$ by the modified variance condition.

The numerical tests show that the algorithm is competitive with other algorithms.

4. Numerical Examples

The examples of this section are quite challenging. One example was proposed by Habetler and Kostreva [2] to illustrate the concept of discontinuous nondegenerate P-mapping. A solution was not provided there. Indeed, the existence of mathematical methods to handle nonlinear equation systems with discontinuous functions was unknown at that time.

The second example is even more elaborate and complex, involving polynomial and trigonometric functions as well as the greatest integer function. It is solved here as a demonstration of the capabilities of the integral global optimization method on nonlinear complementarity problems with a high level of complexity.

EXAMPLE 4.1. Let $f(x_1, x_2) = (-1, -2)^t + h(x_1, x_2)$, where

$$h(x_1, x_2) = \begin{cases} \frac{\sqrt{2}}{2}\begin{pmatrix} 1, & -1 \\ 1, & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, & \text{if } x_1^2 + x_2^2 < 1, \\ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, & \text{if } x_1^2 + x_2^2 \geq 1. \end{cases}$$

For this example the constraint set is

$$S = \{x = (x_1, x_2)^t : x_1 \geq 0, x_2 \geq 0; f_1(x_1, x_2) \geq 0, f_2(x_1, x_2) \geq 0\}.$$

It has a robust partition:

$$S = S_1 \cup S_2,$$

where

$$S_1 = S \cap \{x = (x_1, x_2)^t : x_1^2 + x_2^2 < 1\}$$

and

$$S_2 = S \cap \{x = (x_1, x_2)^t : x_1^2 + x_2^2 \geq 1\}.$$

f_1 and f_2 are relatively robust piecewise continuous. Then we can use discontinuous penalty function to solve the following minimization problem:

$$\min_{x \in S} [x_1 \cdot f_1(x_1, x_2) + x_2 \cdot f_2(x_1, x_2)]. \tag{4.1}$$

As we have expected (4.1) has a unique minimizer $x^* = (1.0, 2.0)^T$ with the global minimum value 0.

EXAMPLE 4.2. Let

$$X = \{(x_1, x_2)^T : x_1, x_2 = 0.001 \cdot j, j = 0, 1, 2, \ldots, 10000\},$$

$$g_1(x) = [1 + (x_1 - x_2 - 1)^2(59 - 26x_1 - 3x_1^2 - 26x_2 + 6x_1x_2 + 3x_2^2] \\ \times [30 + (2x_1 + 3x_2)^2(5 - 20x_1 + 12x_1^2 - 30x_2 + 36x_1x_2 + 27x_2^2)],$$

$$g_2(x) = 10.0\sin^2(\pi x_1) + (x_1 - 1.0)^2[1.0 + 10.0\sin^2(\pi x_2)] + x_2^2$$

and

$$f_1(x) = g_1(x) - [g_1(x)/10]\,, \quad f_2(x) = g_2(x) - [g_2(x)/5]\,,$$

where $[y]$ denotes the integer part of y. The mapping $f = (f_1, f_2)^T : X \to R^2$ is discontinuous and the admissible set is discrete. Let

$$D = \{(z_1, z_2)^T : ([1000 \cdot z_1]/1000, [1000 \cdot z_2]/1000)^T \in X\}\,.$$

It is easy to verify that $D = [0, 10.001) \times [0, 10.001)$ which is robust.

We define a new mapping $F = (F_1, F_2)^T : D \to R^2$, where

$$F_i(z) = f_i([1000 \cdot z_1]/1000, [1000 \cdot z_2]/1000)\,, \quad i = 1, 2\,.$$

For this example the feasible set is

$$S = \{z = (z_1, z_2)^T \in D : z_1 \geq 0, z_2 \geq 0, F_1(z) \geq 0, F_2(z) \geq 0\}\,.$$

F_1 and F_2 are robust piecewise continuous. Then we can use a discontinuous penalty function to solve the following minimization problem:

$$\min_{z \in S} [z_1 \cdot F_1(z_1, z_2) + z_2 \cdot F_2(z_1, z_2)]\,. \tag{4.2}$$

The constrained minimization problem (4.2) has a solution corresponding to a unique minimizer $x^* = (1.0, 0.0)^T \in X$ with the global minimum value 0. After 13 iterations with 670 function evaluations, we obtain

$$x_1 = 1.0\,, \quad x_2 = 0.0\,, \quad F_1 = 0.0\,, \quad F_2 = 3.0\,, \quad v_1 = 0.0\,,$$

where v_1 is the modified variance.

5. Conclusions

In this paper the methodology of integral global optimization is applied to nonlinear complementarity problems under the assumption that the mapping is robust, piecewise continuous, and a nondegenerate P-mapping. Under such weak assumptions, the analysis which arises is the first which can handle these problems. Difficult nonlinear complementarity problems arise in a number of contexts in economics, engineering and management and also may arise as subproblems in system models such as those of constrained parameter estimation and optimal control. Therefore, the contribution of this paper has wide ranging application and, potentially, it may open new avenues of research uniting the subjects of complementarity theory and global optimization.

The examples presented in this paper are illustrative of several noteworthy ideas. Examples 2.1 and 2.2 show that there are solvable nonlinear complementarity problems which are not within the theoretical framework covered here. Examples 4.1 and 4.2, however, are covered by this paper. For these examples, the new solution methodology works remarkably well, making computation seem like an almost routine task. It is our claim that there is no existing methodology which can match that performance.

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